Engineering Notes

Two Self-Test Methods Applied to an Inertial System Problem

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I. Introduction

WHEN using a high-performance costly system, it is clearly worthwhile to develop software that in some sense matches the inherent capabilities of the hardware. Specifically, it is important to consider the development of self-test procedures for the detection of shifts in system parameters and behavior that can have significant effects on system performance. Kalman filtering techniques are often quite useful in monitoring the performance of a system; however, in some cases such a filter is incapable of adjusting to unmodeled phenomena. One reason for this inability is the so-called "oblivious filter" problem, in which the filter gain becomes so small that the filter cannot respond to incoming information. One solution to this problem is to increase the filter gain;[9] however, this in turn leads to an increased estimation error covariance, which cannot be tolerated in certain high-precision systems. A number of alternative methods for self-test system design that avoid this difficulty have been proposed.[1-7] In this paper we consider two of these procedures and adapt them to the problem of estimating the biases in accelerometers and gyroscopes on an inertial platform.

II. An Inertial System Calibration and Alignment Problem

We are given an inertial platform at a fixed, known location on the earth, and we are to calibrate the biases of the accelerometers and gyros mounted on the platform. Our simplified system error model involves 9 state variables which are all given in the coordinates of a reference frame fixed to the platform (P-frame): the 3 platform misalignment errors (x_1, x_2, x_3), representing discrepancies in the alignment of the platform with respect to an inertially fixed reference frame, the 3 accelerometer biases (x_4, x_5, x_6), and the 3 gyro biases (x_7, x_8, x_9). Assuming that the platform is being rotated in a

known manner with respect to inertial space, the dynamics of our model take the form

$$\dot{x}(t) = F(t)x(t) + Q(t)w(t) \qquad (1)$$

$$F(t) = \begin{bmatrix} F_{\psi\psi}(t) & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{\psi\psi}(t) = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \qquad (2)$$

$$Q = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & q_a I & 0 \\ 0 & 0 & q_b I \end{Bmatrix} \qquad (3)$$

Here ω_1, ω_2, ω_3 are the P-frame components of the platform angular velocity with respect to inertial space. We take the initial state covariance $P(0)$ to be a given diagonal matrix.

The available measurements consist of the differences between the actual accelerometer outputs and the outputs that should be produced by gravity at the known platform location:

$$z(t_k) = H(t_k)x(t_k) + v(t_k) \qquad (4)$$

$$H(t_k) = \begin{bmatrix} 0 & -\beta_3(t_k) & \beta_2(t_k) \\ \beta_3(t_k) & 0 & -\beta_1(t_k) \\ -\beta_2(t_k) & \beta_1(t_k) & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad (5)$$

Here (β_1, β_2, β_3) are the P frame components of the one-g vertical specific force vector, and the covariance of the white noise sequence $\{v(t_k)\}$ is given by rI. These measurements are taken at 6 minute intervals. Discretizing Eq. (1), we can design a Kalman filter for the estimation of $x(t_k)$ given the observations of Eq. (4). By processing the 3 components of $z(t_k)$ sequentially, we avoid the necessity of inverting a 3×3 matrix.

In order to obtain accurate platform calibration and alignment, a 6-hr, 3-segment trajectory was devised, in which the platform is rotated at $\rho' = 45°$/hr with respect to the local earth-fixed frame for 2 hr each about the east, north, and vertical axes (in that order; see Ref. 8 for further discussion). Initially the platform 1-, 2-, and 3-axes are north, west, and up, respectively. This leads to the specification of parameters in Eqs. (1-5) given in Table 1. Here t is measurement in hours, Ω_E is earth rotation rate, L is latitude, and $g = 32.2$ (ft/sec²).

For high-precision inertial systems, we require extremely accurate estimation of x; i.e., we want to make the estimation error covariance as small as possible. As discussed in Sec. I,

Table 1   Parameter specification for the system of Eqs. (1-5)

	$0 \le t \le 2$	$2 \le t < 4$	$4 \le t \le 6$
$\omega_1(t)$	$\Omega_E\cos(L-\rho t)$	$\Omega_E\sin L \cos \rho(t-2)$	$\Omega_E\cos L \sin\rho(t-4)$
$\omega_2(t)$	ρ	$\Omega_E\sin L \sin\rho(t-2)$	$\rho+\Omega_E\sin L$
$\omega_3(t)$	$\Omega_E\sin(L-\rho t)$	$-\rho-\Omega_E\cos L$	$-\Omega_E\cos L \cos\rho(t-4)$
$\beta_1(t)$	$g\sin\rho t$	$g\cos\rho(t-2)$	0
$\beta_2(t)$	0	$g\sin\rho(t-2)$	g
$\beta_3(t)$	$g\cos\rho t$	0	0

this leads to an oblivious filter condition, in which the filter is insensitive to small shifts in the state variables. Since extremely small shifts in accelerometer and gyro biases can cause poor overall system performance, we wish to design methods for detecting and compensating for these state jumps.

III.   The WSSR Technique

The first of the two techniques to be discussed is based on a method described in Refs. 1 and 8. Our dynamic model is

$$x(k+1) = \Phi(k+1,k)x(k) + \Gamma(k)w(k) \qquad (6)$$

$$z(k) = H(k)x(k) + v(k) \qquad (7)$$

where the independent, zero mean white noise sequences $\{w(k)\}$, $\{v(k)\}$ have covariances $Q(k)$ and $R(k)$, respectively. We implement a Kalman filter for this system and refer the reader to Refs. 8 and 9 for details. For our purposes, we need only note that the filter generates a measurement residual (innovations) $\gamma(k)$ whose covariance $V(k)$ can be precomputed.

One of the simplest tests for the presence of unmodeled phenomena in the model Eqs. (6) and (7) is the Chi-squared or weighted sum-squared residual (WSSR) test. Our detection criterion is

$$\ell(k) = \sum_{j=k-N+1}^{k} \gamma'(j) V^{-1}(j)\gamma(j) \underset{H_0}{\overset{H_1}{\gtrless}} \epsilon \qquad (8)$$

where H_0 is the hypothesis that system behavior is normal, and H_1 is the hypothesis that a jump has occurred in one of the state variables. If the model in Eqs. (6) and (7) is correct, $\ell(k)$ is a Chi-squared random variable with Np degrees of freedom (dim$\gamma=p$). The size N of the residual "window" and the threshold ϵ are to be chosen to provide an acceptable tradeoff between the probability p_1 of a false alarm (declaring H_1 when actually H_0) and the probability p_2 of a missed alarm (declaring H_0 when actually H_1). The value of p_1 for different values of N and ϵ can be obtained from standard Chi-squared tables. The values of p_2 must be computed for specific times and magnitudes of jumps in state variables. In this case, $\ell(k)$ is a noncentral Chi-squared random variable, and the value of p_2 for different N and ϵ can again be obtained from tables.[4]

As described in Sec. II, for the cal/align problem we have implemented a filter that processes $z_1(k)$, $z_2(k)$, and $z_3(k)$ sequentially. We can then modify the WSSR test by considering the 3 variables

$$\ell_i(k) = \sum_{j=k-N+1}^{k} \gamma_i^2(j)/V_i(j), \qquad i=1, 2, 3 \qquad (9)$$

In this way, we may obtain better jump detection and isolation performance.

An extensive series of simulations has been carried out using the 6-hr trajectory described previously. From the Kalman filter calculations, we obtain the pre-computed bias estimation error variances at the terminal time (6 hr). Jumps

of 10 and 100 times these terminal rms values were inserted at several different times. Several values of N and ϵ were tried, and the values $N=3$, $\epsilon=10.5$ were found to yield a reasonable tradeoff between false alarms, missed alarms, and delay in detection. Ten trials were made for each combination of jump value and time. The number of false alarms, missed alarms, correct detections, and average delay in detection for the 10 σ jump cases are recorded in Tables 2 and 3. Essentially all of the 100 σ jumps were correctly detected with very small delay.[8]

Out of the 480 trials run, only 8 false alarms were observed. Also, we see that WSSR responds more quickly to accelerometer bias shifts than to gyro bias shifts. This is due to the nature of the dynamics—the accelerometer bias enters directly into the outputs z_1, z_2, and z_3 [see Eqs. (4) and (5)], while the gyro biases must first be integrated into misalignments, which then enter the outputs in a time-varying manner. In addition, many of the missed alarms for the jumps occurring early in the 6-hr run should not truly be thought of as false alarms, since early in the run, the Kalman filter is not oblivious and thus it is capable of adjusting to the jumps by itself. Finally, we note that one major drawback of WSSR is its

Table 2   Summary of WSSR detector results with accelerometer jumps

Acc. with jump	Jump time (hr)	Correct detect.	Avg. delay (hr.)	False alarms	Miss alarms
Acc. #1	1	7	0.00	0	3
	2	6	1.18	1	3
	3	8	0.00	0	2
	5	10	0.00	0	0
Acc. #2	1	6	0.48	2	2
	2	3	0.37	0	7
	3	9	0.00	0	1
	5	10	0.00	0	0
Acc. #3	1	3	0.03	1	6
	2	7	0.10	0	3
	3	10	0.54	0	0
	5	9	0.00	0	1

Table 3   Summary of WSSR detector results with gyro jumps

Gyro with jump	Jump time (hr)	Correct detect.	Avg. delay (hr)	False alarms	Miss alarms
Gyro #1	1	5	3.74	0	5
	2	10	2.93	0	0
	3	8	1.74	0	2
	5	10	0.29	0	0
Gyro #2	1	10	1.40	0	0
	2	10	0.87	0	0
	3	2	0.95	0	8
	5	0	...	1	9
Gyro #3	1	2	3.90	0	8
	2	10	0.37	0	0
	3	10	0.28	0	0
	5	10	0.30	0	0

minimal amount of jump isolation capability. Although accelerometer bias jumps are easily isolated, gyro bias jumps are much more difficult. The WSSR as presented is not capable of accurate isolation, as it does not take full advantage of the knowledge of system dynamics. An unsuccessful attempt to utilize information about the observability of the various states at different times in the 6-hr run is reported in Ref. 8.

IV. A Multiple Hypothesis Method

In this section we describe a method developed in Ref. 2 (also see Refs. 6 and 7), which we call the Buxbaum-Haddad (BH) technique. We assume that the actual system model is

$$x(k+1) = \Phi(k+1,k)x(k) \quad \mu(k) \tag{10}$$

where the noise μ has a probability p_0 of being the usual process noise Γw [as in Eq. (6)] and a probability p_i of including additional noise with variance σ_i^2 in the jump direction $f_i (i=1,...,r)$. In other words, the $\{\mu(k)\}$ form a white noise sequence with the density of $\mu(k)$ given by

$$p_0 N(\alpha;0, \Gamma Q \Gamma') + \sum_{i=1}^{r} p_i N(\alpha;0, \Gamma Q \Gamma' + \sigma_i^2 f_i f_i') \tag{11}$$

[here $N(\alpha;m,P)$ is the normal density, with mean m and covariance P, evaluated at α]. It is shown[2] that the conditional density $p(x, k, k)$ for $x(k)$ given $z(1),...,z(k)$, where z satisfies Eq. (7), and $x(0)$ is taken to be normally distributed, is of the form

$$p(x,k,k) = \sum_{i_0=0}^{r} \cdots \sum_{i_{k-1}=0}^{r} p^i N(x;\eta_i, P_i) \tag{12}$$

where $i = (i_0,...,i_{k-1})$ and p^i is the conditional probability that $\mu(0)$ jumps in the i_0 direction, $\mu(1)$ in the i_1 direction, etc., conditioned on $z(1),...,z(k)$ (here we interpret $i_j=0$, as no jump). The mean η_i and covariance P_i are computed by a Kalman filter assuming that $\mu(j)$, $j=1,...,k-1$ has a normal distribution including the term in the f_{i_j} direction. Note that the structure of the optimal filter takes the form of an exponentially growing bank of filters, in which the $i=0$ filter corresponds to the original normal operation filter.

Of course for any practical implementation we must consider approximating the optimal filter. Several techniques are described[2] and we have developed a method that has proven to work quite well in cases in which jumps occur infrequently, as in the cal/align problem. We assume that jumps can only occur at a discrete number of points in time separated by N time steps. At the start of each period of N steps, $(r+1)$ Kalman filters are initiated, one for each jump mode and one for the normal operation hypothesis. Each is initiated with the same mean and covariance; however, in the first step of the period the filters are propagated using the covariance for μ consistent with the corresponding hypothesis. In the remaining steps of the period no jumps are hypothesized in any of the $(r+1)$ filters. At the start of each period, probabilities of each of the hypotheses are initialized with the values $p(i|0) = p_i$, $i=0,1, 1,...,r$. The conditional probability $p(i|k)$ of a jump in the f_i direction is updated according to the formula

$$p(i|k) = \frac{N(z(k);\hat{z}(k|k-1,i),V(k|k-1,i))p(i|k-1)}{\sum_{j=0}^{r} N(z(k);\hat{z}(k|k-1,j),V(k|k-1,j))p(j|k-1)} \tag{13}$$

where $z(k|K-1, i)$ is the one step predicted estimate of $z(k)$ and $V(k|k-1, i)$ is the residual covariance for the filter associated with the ith jump hypothesis. At the end of the N

interval, the various means and covariances are "fused" to form one mean-covariance pair

$$\hat{x}(N|N) = \sum_{i=0}^{r} p(i|N)\hat{x}(N|N,i) \tag{14}$$

$$P(N|N) = \sum_{i=0}^{r} p(i|N) [P(N|N,i) + \hat{x}(N|N,i)\hat{x}'(N|N,i)]$$

$$-\hat{x}(N|N)\hat{x}'(N|N) \tag{15}$$

and the probabilities are reset to their initial values. The $(r+1)$ filters are then reset with the data in Eqs. (14) and (15) and the next N step propagation is begun.

If during the period of N steps, one of the $p(i|k)$ for $i \neq 0$ becomes large, we declare that a jump has occurred. If N is sufficiently long and if the jump is observable over the period, the probability associated with the actual jump will approach 1. The choice of N is clearly of great importance in system performance. If N is too short, the system does not have enough time to look at the data to find the failure before the fusing and resetting is performed. If N is too long, although one will correctly detect a jump if it actually occurred at the start of the N interval, one might not be able to detect jumps that occur in the middle of the interval, as all $(r+1)$ filters become oblivious.

Several runs have been made for the cal/align problem. The 6 failure directions $f_1,...,f_6$ correspond to jumps in the biases (states $x_4,...,x_9$). The values of $\sigma_1,...,\sigma_6$ were taken to be 100 times the terminal rms error variances in $x_4,...,x_9$, respectively. The $p_i, i=1,...,6$ were taken to be 0.005, $p_0=0.97$, and $N=5$. Table 4 summarizes the performance of the BH system on 2 runs. The table contains the probabilities $p(i|N)$ of each hypothesis at the end of the N intervals (just before reset). Only the non-negligible probabilities are displayed.

The first set of data comes from a run in which a jump of 10 σ occurs in x_5 just after 3 hr. Noting that $p(2)$ corresponds to a jump in x_5, and that 3.5 hr is the first point in the table at which we could possibly see the effect of the jump, we see that the BH filter works remarkably well. The second set of data comes from a run with a 10 σ jump in x_7 just after 3 hr. In this case $p(4)$ corresponds to a jump in x_7, and $p(5)$ to a jump in x_8. Note that, as expected, there is some delay in the detection of gyro jumps, and there is some ambiguity, as each gyro affects several misalignments, which in turn affect several measurements. The data in Table 4 points out the difficulty in detecting gyro jumps and also indicates a "self-correcting" aspect of the BH system. At first, the system is unsure which of the two gyros caused the observed output irregularity, and by increasing both $p(4)$ and $p(5)$ the system effectively increases the overall filter gain for both gyro estimates. The x_7 estimate is then corrected, but, since x_8 did not shift, its estimate is degraded. During the following N interval, the BH detects its own error in degrading the estimate of x_8 as a jump

Table 4   Conditional probability time histories (BH system)

Time (hr)	10 σ jump in x_5 at 3 hr		10 σ jump in x_7 at 3 hr		
	$p(0)$	$p(2)$	$p(0)$	$p(4)$	$p(5)$
0.5	0.97	0.01	1.00	0.00	0.00
1.0	1.00	0.00	1.00	0.00	0.00
1.5	1.00	0.00	1.00	0.00	0.00
2.0	1.00	0.00	1.00	0.00	0.00
2.5	1.00	0.00	1.00	0.00	0.00
3.0	1.00	0.00	1.00	0.00	0.00
3.5	0.00	1.00	0.99	0.00	0.01
4.0	0.98	0.02	0.00	0.54	0.46
4.5	1.00	0.00	0.54	0.05	0.41
5.0	1.00	0.00	0.99	0.00	0.01
5.5	1.00	0.00	1.00	0.00	0.00

in x_8. In this manner the x_8 gain is kept large a while longer, allowing the filter to correct its mistake.

Thus we see that the BH method performs exceptionally well and is quite useful in pointing out important aspects of the dynamics of jump detection. It has the advantage of directly providing jump isolation information [the $p(i)$], but suffers from computational problems. In general we must implement $(r+1)$ n-dimensional filters with covariances computed on-line. The BH method is still useful in providing a benchmark to which one can compare other methods. In addition, the intial success we have had with this approach indicates that serious consideration should be given to finding computationally feasible adaptations of this sytem. One possibility is to use the WSSR to detect state jumps and then to switch to the BH system for jump isolation and estimate compensation. In this manner, we do not increase the computational load until after a jump is detected. This dual-mode method will introduce delays in isolation and compensation, but it is anticipated that for the cal/align problem this will not be a significant problem. For other problems, one may have to re-evaluate the tradeoff between system performance and computational complexity.

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Observer Algorithm Identification of System Structure, Parameters, and States

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I. Introduction

THE problem considered here is the estimation of the structure, states, and parameters of a nth order linear time-invariant plant where only the input and

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output can be observed. This development is an extension of the results of Lion.[1,7] Lion considered parameter identification without state estimation. Luders[3] related the "observer"[2] state estimations structure to Lion's algorithm resulting in a state and parameter estimation algorithm. The observer formulation presented here has the advantage of ability to identify the unknown system order. In addition this formulation has greater design freedom in "state variable filter" selection than that given in Ref. 3.

II. Brief Development

It is assumed that the completely observable system can be described by an nth order time-invariant vector differential equation. The order n of the unknown system may be determined by this algorithm. For the sake of simplicity, the new canonical form is derived only for the single-input single-output case. Nevertheless the extension of this canonical form to the multi-input case is straightforward.[4]

Given a stable stationary observable system transfer function with unknown parameters α, β

$$G(s) = \frac{\sum_{1}^{n+1} \beta_i s^{i-1}}{\sum_{1}^{n+1} \alpha_i s^{i-1}} \tag{1}$$

Find a convergent parameter and state estimator. Solution: restriction of Lion's[1] "state variable filter" to a simple form leads to a state estimate (observer) relationship. The transfer function (one) can be expressed more conveniently in terms of *known parameters* $\lambda_i > 0$ as follows

$$G(s) = \frac{\sum_{1}^{n+1} M_i b_i}{\sum_{1}^{n+1} M_i a_i} \tag{2}$$

where

$$a_1 = 1, \quad M_{n+1} = 1, \quad M_i \underset{i \neq n+1}{\triangleq} \prod_{j=1}^{n} \frac{1}{s + \lambda_j}$$

In expression (2) the (a, b) are now the parameters to be identified. The transformation relating (a, b) to (α, β) involving (λ) can be derived easily by equating coefficients of like powers of s. Note that if all $\lambda_i = 0$ then $(a, b) = (\alpha, \beta)$. The form of Eq. (2) is motivated by state estimate convergence requirement.

The new canonic form is as follows

$$\dot{w} = \Lambda w + hu \tag{3}$$

$$\dot{v} = \Lambda v + hy \tag{4}$$

$$y = \left[\frac{1}{a_{n+1}} \right] [b^T w + b_{n+1} u - \underline{a}^T v] \tag{5}$$

where

$$b^T = (b_1, \ldots, b_n), \quad \underline{a}^T = (1, a_2, \ldots, a_n)$$

$$h^T = (0, \ldots 0, 1)$$

$$\Lambda = \begin{bmatrix} -\lambda_1 & 1 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & -\lambda_n \end{bmatrix}$$

u = system input; y = system output